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Percolation of the Loss of Tension in an Infinite Triangular Lattice

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We introduce a new class of bootstrap percolation models where the local rules are of a geometric nature as opposed to simple counts of standard bootstrap percolation. Our geometric bootstrap percolation comes from rigidity theory and convex geometry. We outline two percolation models: a Poisson model and a lattice model. Our Poisson model describes how defects—holes is one of the possible interpretations of these defects—imposed on a tensed membrane result in a redistribution or loss of tension in this membrane; the lattice model is motivated by applications of Hooke spring networks to problems in material sciences. An analysis of the Poisson model is given by Menshikov *et al.*[(4)] In the discrete set-up we consider regular and generic triangular lattices on the plane where each bond is removed with probability $1-p$. The problem of the existence of tension on such lattice is solved by reducing it to a bootstrap percolation model where the set of local rules follows from the geometry of stresses. We show that both regular and perturbed lattices cannot support tension for any $p<1$. Moreover, the complete relaxation of tension—as defined in Section 4—occurs in a finite time almost surely. Furthermore, we underline striking similarities in the properties of the Poisson and lattice models.

KEY WORDS: Equilibrium tension; self-stress; spider web; triangular lattice; percolation on graphs; bootstrap percolation; graph rigidity.

1. INTRODUCTION

Consider a planar tensed membrane in space clamped on its boundary. What happens when holes are created in this structure? When will it still support tension? When will there be floppy portions that bend and flex? When will the whole structure become floppy with tension vanishing throughout the membrane? Naturally this depends on how the holes are

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distributed and just what the structure of the membrane is. We present two classes of percolation models, where tension can exist in a natural sense, and where the creation of holes can have the consequence of relieving the tension. One approach is a continuous bootstrap-like percolation of compact defects distributed with a Poisson Law. The other is a bootstrap percolation on a triangular lattice. In both of these models it is the geometric properties of the underlying structure (after the holes are created) that determines whether or not the tension exists. Thus, our paper introduces a new class of geometric bootstrap percolation models, where the local rules are determined via the convexity property. Namely, in the continuous model described in Section 2 the local rules force the removal of all convex corners (angles). In the discrete model our local rules (see Figs. 10–11) prescribe the removal of the edges that are incident to the vertices whose stars are convex. Schonmann,[1,2] Mountford[3] and others earlier studied various bootstrap percolation models where local rules are defined via the combinatorics (geometry) of local configurations.

We prove here that an infinite triangular (regular or perturbed) lattice, where each edge has been removed independently with probability $1-p > 0$, cannot support an equilibrium tension almost surely (a.s.—throughout the text). There are strong parallels between this result and the results obtained by Menshikov *et al.*[4] for our continuous model. In the continuous model the positions of numerous holes are distributed homogeneously in the plane according to a Poisson Law with $\lambda > 0$, and their shapes are independently identically distributed (i.i.d.—throughout the text) random functions on a circle which are independent of the Poisson Process. As with the lattice model, tension vanishes almost surely. But in this model, the criterion for tension existence is that there is some triangulation of the complement of the holes such that an equilibrium tension is supported in its 1-skeleton.

In our definition, stress (and, in particular, tension) is a real scalar quantity $\omega_{ij} = \omega_{ji}$ associated to each edge (i, j) between vertex i and vertex j of a graph underlying a framework that triangulates the region in the plane. This stress is said to be an equilibrium stress if the vector sum $\sum_j \omega_{ij}(\mathbf{v}_\mathrm{i} - \mathbf{v}_\mathrm{j}) = 0$ for each vertex $\mathbf{v}_\mathrm{i}$ (treated as a vector in Euclidean space) of the graph other than pinned vertices. There is no equilibrium condition for pinned vertices. If an edge (i, j) has $\omega_{ij} > 0$, it is said to be in tension. When a framework is connected and has all of its edges in tension it is easy to show that this framework is rigid. This is one of the main tools to show rigidity and one of the main reasons that the existence of an equilibrium stress with all of its members in tension is of interest here. But the stress, as it is defined here, is more accurately thought of as a stress coefficient, rather than what might be usually referred to as a stress in physics or engineering. Each ω_{ij} is not a force by itself. The vector quantity $\omega_{ij}(\mathbf{v}_\mathrm{i}-\mathbf{v}_\mathrm{j})$ is essentially the physical force involved. But the ω_{ij} are more easily dealt with mathematically, and they are what has been used in the mathematical literature.

When the percolation process of edge removal or hole creation is performed, for any particular graph (the lattice model) or complement of the holes (the Poisson model), the determination of whether there exists an equilibrium stress that is positive on all the edges of some graph can be difficult to determine. Fortunately, however, to calculate the critical tension threshold for the Poisson model it is enough to consider only the situation where convex holes intersect. If there is a region in the plane that is removed, creating a hole that is not convex, then the convex hull of a connected component has no tension in its interior. We call such an area *defective*. It turns out that with high probability, these holes coalesce into defective areas that and eventually cover the entire plane in the infinite case. The proof of this is one of the main points of the paper by Menshikov *et al.*[4]

In Section 1.1 we carefully define the notions of stress and rigidity. In Section 1.2 we show how the discrete approach based on these notions can be applied to help understand the rigidity of regions in the plane. Section 2 treats the continuous membrane model and recent probabilistic results for it. The main probabilistic result of this paper states that the relaxation of tension on a triangular lattice (regular or slightly perturbed) where each edge has been removed independently with probability $1-p>0$ occurs in a finite time (discrete time for our bootstrap process is defined in Section 3) almost surely; this is proved in Section 3. Menshikov *et al.*[4] showed that in the continuous case the relaxation of tension also occurs in a finite time a.s. In Section 3 we analyze similarities between the processes of tension relaxation for the continuous and discrete cases. In addition, we conjecture that our methods used for triangular lattices can be applied to a broader class of planar graphs. In the last section we discuss connections between the problem of tension percolation for the infinite regular triangular lattice and the same problem for finite subgraphs of this lattice.

Both models assume that after the edge removal or hole creation the remaining medium remain static. If the medium, lattice or membrane, has the property that it can deform and recreate another stressed configuration after the removal of the edges (the lattice model) or the holes creation (the continuous model), it could arrive at a new stressed configuration in equilibrium. If the medium has such elastic properties, both results should rather be interpreted not as immediate relaxation, but as an inevitable displacement restoring the ability of the system to support tension; in other words, to preserve strong stability, the system has to rearrange itself. For

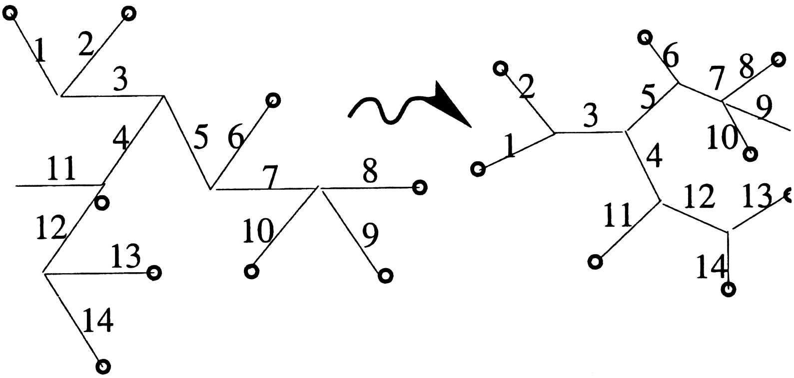

Fig. 1. A perturbed fragment of a sub-graph of the triangular lattice; vertices marked with circles may be adjacent to other edges of the sub-graph.

example, in the lattice model a star of ϵ-type (see Fig. 10 in Section 3) can reshape into the star that can support tension (see Figs. 1 and 2). In the continuous model the displacement can be pictured in many different ways. Of course, if the probability distribution is such that the medium is not even connected, then the material will not even be able to rearrange itself after the edges have been cut or the holes have been created. Note, that unlike tension percolation, this connectivity percolation has a critical probability value below which the medium remains connected, almost surely, and above which it is disconnected, almost surely (Menshikov, Sidorenko 1987).

Using computer simulation Tang and Thorpe[5,6] studied the rigidity of large finite networks of elastic springs of natural length zero, where tension is supplied by the boundary frame. They refer to this model as a *network of elastic springs under tension.* In their model as soon as bonds (edges) are removed the network (framework) deforms around the missing edges and the sites move to new equilibrium positions. "For the perfect

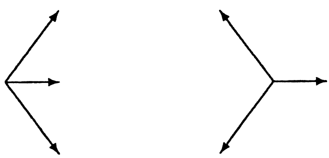

Fig. 2. An ϵ-star can reshape into a star supporting tension.

lattice where no bonds are missing, the lattice node serves as the equilibrium position because the forces exerted on a site through adjacent stretched springs balance each other. However, as soon as bonds are removed, this balance is destroyed and the NETWORK DEFORMS AROUND THE MISSING SPRINGS AND THE SITES MOVE TO NEW EQUILIBRIUM POSITIONS. It is around these new equilibrium positions which can be very far away from the corresponding original lattice nodes, that the sites in the network are vibrating..." Their simulation results suggest that there is a non-trivial threshold that lies between the critical thresholds for rigidity percolation (TRP) and connectivity percolation (TCP). The value of this threshold depends on the value of the external force. As the external force changes from zero to infinity, the value of the threshold changes from TRP to TCP.

We observe that our results do not contradict those of Tang and Thorpe for the reasons mentioned in the preceding paragraphs.

1.1. Frameworks: Rigidity and Stresses

A bar-and-joint framework is a graph (possibly, with countably many vertices) together with its realization in $\mathbb{R}^d$. *We consider only discrete frameworks: any compact subset of $\mathbb{R}^d$ may contain only a finite number of vertices of framework.* Denote by $G(E, V, V_0; \mathbf{p})$ a framework in $\mathbb{R}^d$ with the edge set E, and the vertex set V with pinned (fixed in $\mathbb{R}^d$) subset of vertices $V_0 \subset V$; here $\mathbf{p}$ is the list of all the coordinates of the vertices of the framework. We will denote the graph of the framework by $G(E, V, V_0)$, where V_0 is the set of vertices that must be pinned in a realization. Thus, in our notation $\mathbf{p}$ defines a realization of the graph $G(E, V, V_0)$ in $\mathbb{R}^d$. Vertices that are not pinned are called *free*. If $V_0 = \varnothing$, we will write simply $G(E, V; \mathbf{p})$. Notice, that in the mathematics of rigidity there is a tendency to use term *framework* instead of *network* preferred by physicists. Denote by $\mathbf{v}_i$ the vector of coordinates of vertex $v_i \in V$.

Definition 1.1. An equilibrium stress (or self-stress) is an assignment of real numbers $\omega_{ij} = \omega_{ji}$ to the edges, a tension if the sign is positive, or a compression if the sign is negative, so that the equilibrium conditions

$$\sum_{\{j \mid (ij) \in E\}} \omega_{ij}(\mathbf{v}_j - \mathbf{v}_i) = 0$$

hold at each vertex $\mathbf{v}_i \in V \setminus V_0$ (see Fig. 3).

Definition 1.2. A framework $G(E, V, V_0; \mathbf{p})$ that has an equilibrium stress, positive on all edges, is referred to as a *spider web*.

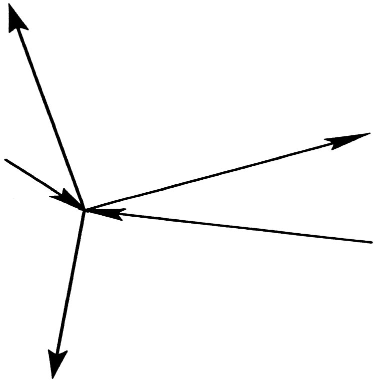

Fig. 3. Equilibrium stress.

For example, an infinite regular triangular lattice is, obviously, a spider web (see Fig. 4 for a piece of such grid). The following propositions immediately follow from the definition of stress.

Proposition 1.3. If $G = (E, V; \mathbf{p})$ is a spider web in $\mathbb{R}^d$, then G has infinitely many edges and vertices, and their convex hull is an affine subspace of $\mathbb{R}^d$.

Proposition 1.4. If $G(E, V; \mathbf{p})$ is a spider web, then for each vertex $\mathbf{v}$ of G the convex hull of the vertices adjacent to $\mathbf{v}$ contains $\mathbf{v}$.

Denote the set of vertices adjacent to $\mathbf{v}$ by $A(\mathbf{v})$. The following proposition follows from the definition of spider web.

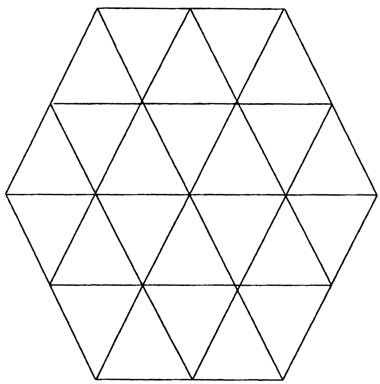

Fig. 4. Triangular grid.

Proposition 1.5. Let $\mathbf{v}$ be a vertex of a spider web $G(E, V; \mathbf{p})$ in $\mathbb{R}^d$. Suppose there is a subset of vertices of $A(\mathbf{v})$ such that its affine hull is a hyperplane in $\mathbb{R}^d$ passing through V. Then, if the affine hull of $A(\mathbf{v})$ is $\mathbb{R}^d$, the convex hull of $A(\mathbf{v})$ intersects both open half-spaces determined by this hyperplane.

Two frameworks in $\mathbb{R}^d$ are called *edge equivalent* if they have the same graph and the same lengths of all edges. Two edge equivalent frameworks in $\mathbb{R}^d$ are called *congruent* if all distances between corresponding pairs of vertices are the same. Notice, that for a finite framework the list of the vertex coordinates $\mathbf{p}$ can, obviously, be regarded as a point in the space of parameters $\mathbb{R}^{d|V|}$.

Definition 1.6. A finite framework $G(E, V, V_0; \mathbf{p})$ in $\mathbb{R}^d$ is called *rigid* in $\mathbb{R}^d$ if there is a neighborhood $N_{\mathbf{p}} \subset \mathbb{R}^{d|V|}$ of $\mathbf{p}$ such that any other realization $\mathbf{q}$ of graph $G(E, V, V_0)$ satisfying the following conditions (1)–(3) is congruent to G.

(1) *$G(E, V, V_0; \mathbf{q})$ is edge equivalent to $G(E, V, V_0; \mathbf{p})$,*

(2) $\mathbf{q} \in N_{\mathbf{p}}$,

(3) *the pinned vertices of $G(E, V, V_0; \mathbf{q})$* coincide with *the pinned vertices of $G(E, V, V_0; \mathbf{p})$.*

If a framework G satisfies the above definition with $N_{\mathbf{p}} = \mathbb{R}^{d|V|}$, it is called *globally rigid* in $\mathbb{R}^{d|V|}$. Note that a globally rigid framework is automatically rigid. A framework that is not rigid is called *flexible*. It is important to specify the dimension of the space where our framework G is considered. A graph can be rigid in $\mathbb{R}^2$, but not rigid in $\mathbb{R}^3$: for example, the graph depicted in Fig. 5 has motions that keep the boundary vertices on the plane, but move the vertices U, V and W, lying inside the triangle, from the plane (dashed lines show that the extensions of the edges do not have a common point: this is a sufficient condition for this graph to be flexible in the space). There are a few ways to define rigidity for infinite graphs, but the existing tools of rigidity theory allow one to work only with those definitions where the rigidity of an infinite graph is understood as the rigidity of its finite subgraphs. It is natural to refer to this type of rigidity as *finite* rigidity (see Bezdek *et al.*[7]). Since in this paper we deal only with finite types of rigidity we shall omit the word *finite* throughout the rest of the paper.

In the following definition all frameworks are assumed to have no pinned vertices.

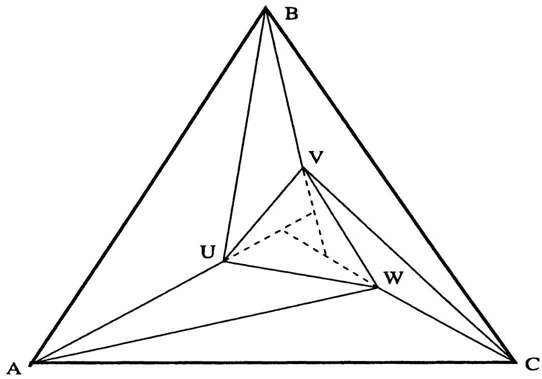

Fig. 5. Rigid in the plane, but not in space.

Definition 1.7. An infinite framework $G(E, V; \mathbf{p})$ in $\mathbb{R}^d$ is referred to as *rigid* if any finite sub-framework of $G(E, V; \mathbf{p})$ is contained in a rigid finite sub-framework of $G(E, V, \mathbf{p})$.

The above definition of rigidity was adopted by Holroyd[8, 9] in his studies of generic rigidity percolation on lattices (see also Grimmett[10]). Since the definition of rigidity for finite graphs can be applied to infinite graphs with all but finitely many vertices pinned, the following definition of pseudorigidity is consistent.

Definition 1.8. An infinite framework $G(E, V, V_0; \mathbf{p})$ in $\mathbb{R}^d$ is referred to as (globally) *pseudorigid* if for any finite subset V' of V the framework obtained from $G(E, V, V_0; \mathbf{p})$ by pinning all of the vertices in $V \setminus V'$ is (globally) rigid.

To avoid confusion, let us notice that in some papers (e.g., Bezdek *et al.*,[7] and Connelly[11]) the above property is called finite rigidity. Rigidity in $\mathbb{R}^d$ obviously, implies pseudorigidity in $\mathbb{R}^d$. However, rigidity in $\mathbb{R}^2$ does not imply pseudorigidity in $\mathbb{R}^3$. For example, an infinite graph triangulating $\mathbb{R}^2$ is always rigid in the plane, however, if it has a subgraph shown in Fig. 5, it is not pseudorigid in space. Let us illustrate the differences between rigidity and pseudorigidity: the regular triangular lattice is rigid in $\mathbb{R}^2$ and pseudorigid in $\mathbb{R}^3$, but not rigid in $\mathbb{R}^3$, whereas the square lattice is pseudorigid in $\mathbb{R}^3$, but not rigid in $\mathbb{R}^d$ $(r \geqslant 2)$. The pseudorigidity of the square lattice follows from the basic properties of spider webs (see Connelly[12, 13]).

Our motivation for introducing tension percolation models was to study the properties of random graphs that guarantee the rigidity not only in $\mathbb{R}^2$, but also in $\mathbb{R}^3$. One of such properties is the existence of an equilibrium tension (Connelly,[12] and Connelly and Whiteley[14]). An infinite framework can be rigid in $\mathbb{R}^2$ but not even pseudorigid in $\mathbb{R}^3$. For example, let ABC be a triangle in the regular triangular lattice. Now, add a triangle UVW and edges BV, AU, and CW to the lattice, as it is shown in Fig. 5. The resulting infinite graph will still be rigid in the plane, but not in 3-space, since the added vertices can be lifted from the plane without changing the lengths of the edges. The computer simulation program of Jacobs and Thorpe[15, 16] constructs large rigid clusters (finite, indeed) by pseudorandom edge removal from the triangular lattice; it is interesting that most of these clusters are rigid in $\mathbb{R}^2$, but flexible in $\mathbb{R}^3$ with the boundary pinned. Our main result explains, to some extent, why these clusters should not be rigid in $\mathbb{R}^3$ with the boundary pinned: a spider web is always pseudorigid (Proposition 1.11), but for a triangular lattice $\mathbb{T}$ any non-negligible edge removal has the consequence that no infinite subset of $\mathbb{T}$ is a spider web (Theorem 3.2).

The rigidity and plasticity properties of a glass are related to how amenable the glass is to continuous deformations requiring little energy. From a physical point of view it is not enough to declare that the distance constraints force the structure to have only one configuration, since the bonds in a physical network do not behave as ideal bars in a framework. There should be a way of describing the behavior of the system as it is perturbed. That is why physicists often consider the energy function defined on the edges of a network of Hooke springs: each spring has some optimal length at which its energy is minimal, stretching or shortening a spring increases the energy of this connection. A *tensegrity* framework is a generalization of this model where besides Hooke springs there are members whose energy increases with the distance, and members whose energy decreases with the distance. In context of energy considerations it is often useful to work with the notion of tensegrity framework (Roth and Whiteley,[17] and Connelly and Whiteley[14]).

In a tensegrity framework all edges are partitioned into three types, cables E_+, struts E_-, and bars E_0, i.e., $E = E_0 \cup E_+ \cup E_-$. Together, struts, cables, and bars are called members. If a cable is stretched, the energy in the cable increases; if a strut is shortened, the energy in it increases too. Any change in the length of a bar forces the energy to increase. Therefore, networks of Hooke springs are bar tensegrities from a mathematical point of view.

Let $G(E_0, E_+, E_-; V, V_0; \mathbf{p})$ be some tensegrity framework in $\mathbb{R}^d$. The energy $\mathfrak{H}_{ij}$ of member (ij) considered as the function of its squared length l_{ij}^2

— is monotone increasing if (ij) is a cable,

— is monotone decreasing if (ij) is a strut,

— has a strict local minimum at l_{ij}^0 called the equilibrium length of (ij), if (ij) is a bar.

It is natural to define the energy function $\mathfrak{H}$ of a finite tensegrity framework (finite network of Hooke springs) as the sum of the energy functions of its members. Thus,

$$\mathfrak{H} = \tfrac{1}{2} \sum_{(ij) \in E} \mathfrak{H}_{ij}(|\mathbf{v}_j - \mathbf{v}_i|^2) = \tfrac{1}{2} \sum_{(ij) \in E} \mathfrak{H}_{ij}(l_{ij}^2). \tag{1}$$

When all members are bars the simplest way to define the energy function is as follows

$$\mathfrak{H} = \tfrac{1}{2} \sum_{(ij) \in E} a_{ij}(l_{ij} - l_{ij}^0)^2, \tag{2}$$

where the sum is over all ordered pairs of vertices of the framework, l_{ij} is the length of the bond between i and j, l_{ij}^0 is the equilibrium bond length, and $a_{ij} > 0$ is the spring constant of the bond between vertices v_i and v_j. Here $\mathfrak{H}_{ij}(x) = x + (l_{ij}^0)^2 - 2\sqrt{x}\, l_{ij}^0$.

In the spirit of the definition of equilibrium stress we assume that a strut can support only compression, a cable can support only tension, and a bar can be under either type of stress, depending on whether its length is larger or smaller than l_{ij}^0. For more detailed information on tensegrities see the works of Roth and Whiteley,[17] Connelly and Whiteley,[14] and Connelly.[12, 13]

Definition 1.9. A finite tensegrity framework $G(E, V, V_0; \mathbf{p})$ in $\mathbb{R}^d$ with pinned vertices $V_0 \subset V$ is called *prestress stable* if

(1) The first derivatives of $\mathfrak{H}_{ij}(x)$ evaluated at $x = |\mathbf{v}_i - \mathbf{v}_j|^2$ constitute an equilibrium stress on G.

(2) the second differential of $\mathfrak{H}(|\mathbf{v}_i - \mathbf{v}_j)|^2)$—regarded as the function of the coordinates of point $\mathbf{p} \in \mathbb{R}^{d|V|}$—is a positive semidefinite quadratic form whose kernel, restricted to infinitesimal motions leaving V_0 unmoved, consists of trivial infinitesimal motions of the framework.

As in the case of rigidity, this definition can be applied to infinite frameworks with only finitely many free vertices—and we use this in the following definition.

Definition 1.10. An infinite framework $G(E, V, V_0; \mathbf{p})$ in $\mathbb{R}^d$ is called *prestress stable* if for any finite subset V' of V there is an interpretation of the edges which have at least one vertex in V' as either cables or struts such that the *tensegrity* framework obtained from $G(E, V, V_0; \mathbf{p})$ by pinning all of the vertices in $V \setminus V'$ is *prestress stable* in $\mathbb{R}^d$.

The concept of prestress stability comes from engineering and, basically, accounts for local minima of the energy function. This concept is defined in Connelly[12] and Connelly and Whiteley.[14] If $\mathfrak{H}_{ij}$ are twice continuously differentiable, a prestress stable framework realizes a local minimum of the global energy function $\mathfrak{H}$ (Connelly and Whiteley[14]). Note that if $\mathfrak{H}_{ij}$ are defined by formula 2, they are twice continuously differentiable on $(0, \infty)$.

1.2. Tension

The existence of a tension (a positive equilibrium stress) on a framework in the plane implies some important rigidity properties for this framework considered in three-space. This may have some interesting consequences for modeling physical properties of materials with networks of Hooke springs and geometry of convex surfaces. The rigidity properties of infinite graphs (lattices) drew the attention of physicists since the early 80's. It turns out that real glasses are well represented by random central-force networks of Hooke springs (Thorpe[18]). The success of these methods resulted in good characterization of elastic properties of glasses like $Ge_xAs_ySe_{1-y}$ (Thorpe[18]). In their experiments the variation of the parameters x and y is directly linked to variation of the probability of edge removal in the independent rigidity percolation model on the triangular lattice. The rigidity analysis of random networks has also been used for modeling physical properties of proteins, polymers and semiconductors (Thorpe and Duxbury[19]).

Let G be an infinite framework rigid in $\mathbb{R}^2$. The example discussed in Section 1 clearly demonstrates that G need not be rigid in $\mathbb{R}^3$. The pictures produced by Jacobs and Thorpe's program[15, 16] also give examples of $\mathbb{R}^2$-rigid, but not $\mathbb{R}^3$-rigid pseudorandom graphs. Meanwhile, the spider web property implies the pseudorigidity in $\mathbb{R}^d$ for any $d > 1$.

Proposition 1.11. Let $G(E, V, V_0; \mathbf{p})$ be a (possibly infinite) spider web in $\mathbb{R}^2$ with pinned vertices $V_0 \subset V$. Then

(1) $G(E, V, V_0; \mathbf{p})$ is globally pseudorigid in $\mathbb{R}^d$ $(d > 1)$;

(2) $G(E, V, V_0; \mathbf{p})$ is prestress stable in $\mathbb{R}^d$ $(d > 1)$.

These results can be derived from Connelly[12] and Connelly and Whiteley,[14] where they are proved for finite frameworks (see also Connelly[12, 13]). The proofs directly apply to infinite frameworks, since the pseudorigidity has been defined via finite subgraphs of G.

2. POISSON MODEL

2.1. Tension in a Membrane

Let M be a tensed membrane (film) clamped on its boundary. A small convex hole made in the membrane results in the redistribution of tension in the rest of the membrane. Clearly, if we have a non-convex hole (which can also be interpreted as the union of a number of convex overlapping holes) tension ought to vanish on the convex hull of this set (see Fig. 6).

It is, however, less intuitive that tension may vanish at some subset of the complement of a collection of convex non-overlapping holes. For example, the convex hull of three holes shown on Fig. 7 cannot support tension; this can even be verified with a sheet of some elastic material and scissors. Therefore, if the area where tension vanishes is interpreted as defective, all three polygons on Fig. 7 ought to coalesce into one big defect. A mathematical explanation of this *coalescence effect of a "pinwheel configuration"* is given in Menshikov *et al.*[4] Roughly speaking, the non-existence of tension on the convex hull of the three holes is due to the visible "swirl" in the area where the triangles "almost" meet.

By a (convex) tiling of a closed planar set with piecewise-linear or no boundary we mean a *locally finite* partition of this set into subsets of three types: open convex polygons called 2-cells, open segments called edges or

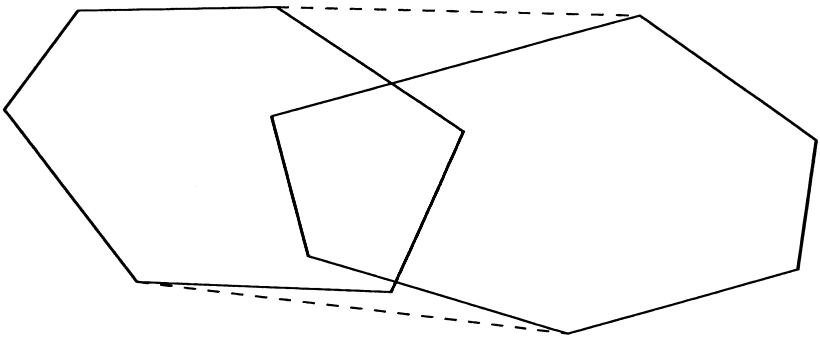

Fig. 6. Two overlapping holes: tension must disappear on the convex hull of them.

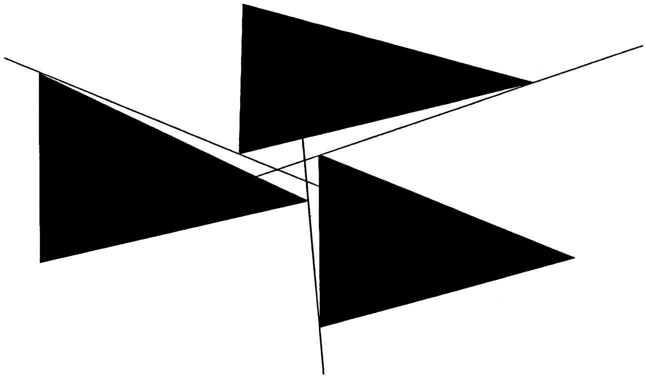

Fig. 7. Three non-overlapping holes: tension must disappear on the convex hull of them.

1-cells, and points called vertices or 0-cells. The 1-skeleton of a tiling is a framework whose vertex set is the vertex set of the tiling, and whose edge set is the tiling's edge set. A triangular tiling where any two triangles whose closures intersect can only make contact either at a common vertex or at a full common edge is called a *triangulation* (see Fig. 8).

Definition 2.1. Let M be a set with a polygonal or no boundary in $\mathbb{R}^2$ (M might be all of $\mathbb{R}^2$), and let $\mathscr{H}$ be a collection of open polygons in M, such that the number of polygons intersecting any compact subset of $\mathbb{R}^2$ is finite. We call the elements of $\mathscr{H}$ *holes* and denote by H the pointwise union of the holes.

Definition 2.2. In the context of the above definition we say that $M \setminus H$ supports tension if $M \setminus H$ admits a partition with the edge set E and

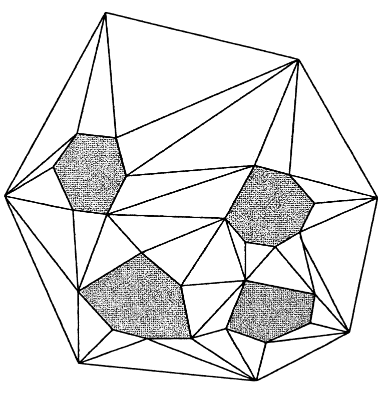

Fig. 8. A triangulation of the complement of the shaded area in the septagon.

vertex set V such that the framework $(E, V, V \cap \partial M)$ is a spider web. Let S be a closed subset of $M \setminus H$. We say that tension is lost on S if there is no closed subset A of $M \setminus H$ such that A supports tension and contains S.

Evidently, in this definition a general convex tiling can be replaced by a triangulation without any loss of generality. A direct generalization of this definition to the case of higher dimensions is possible, but not quite natural, since not all spider webs in dimensions higher than 2 can be interpreted as 1-skeletons of polyhedral tilings (see Connelly and Whiteley[14]). In the planar case the situation is simplified by the fact that any spider web with self-intersections can be turned into the 1-skeleton of a polygonal tiling by adding points of self-intersections to the vertex set of the framework, and modifying the edge set accordingly: the cone of positive stresses of the 1-skeleton of the new partition contains the cone of positive stresses of the original skeleton. A more natural definition for the higher dimensions would be one in which we require the existence of a three-dimensional spider web in the complement of the holes such that each vertex of each hole is incident to at least one edge of the web.

Let us now make some observations about holes. First, if a hole is non-convex, then there is no triangulation of the complement such that its 1-skeleton (vertices on the boundary of M are pinned, indeed) supports a non-zero equilibrium tension. For instance, the equilibrium of forces at vertex **v** in Fig. 9 is impossible, if all edges incident to this vertex are under tension. Therefore, if two holes overlap, and their union is not convex, such as in Fig. 6, tension vanishes on all of their convex hull. Intuitively, the vanishing of tension is rather obvious—the pieces ABC and EFG are "floppy" in the space. This is called the *coalescence effect of overlapping holes*. We summarize this observation in the following proposition.

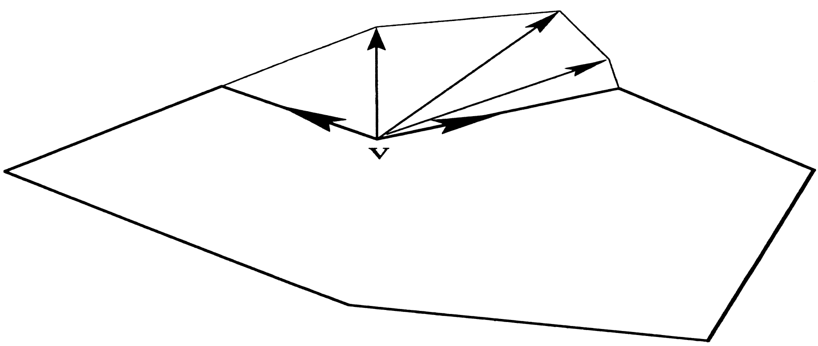

Fig. 9. Non-convex hole.

Proposition 2.3. Let H be a an open polygonal subset of $\mathbb{R}^2$. If $\mathbb{R}^2 \setminus H$ supports tension, all connected components of H are convex polygons.

Notice, that the above proposition is not valid for dimensions greater than two. Obviously, a saddle point of a 3-dimensional hole can serve as a vertex of a spider web realized in the complement of the hole. Nevertheless, by Proposition 1.5 a set supporting tension in $\mathbb{R}^d$ cannot have points of strict convexity.

Proposition 2.4 (Menshikov *et al.*[(4)]). Let M be a convex subset of $\mathbb{R}^2$ with a polygonal or no boundary (M might be all of $\mathbb{R}^2$). For a *finite* set of polygonal holes $\mathscr{H}$ there is a supporting tension subset $S_{\max}$ of $M \setminus H$ such that any subset of $M \setminus H$ supporting tension is contained in $S_{\max}$.

Thus, when the number of holes is finite, $M \setminus H$ can be partitioned into two polygonal subsets, the unique *maximal* (with respect to inclusion) subset supporting tension and its complement where tension vanishes. The case of infinite system of holes is more complicated. Even under additional restrictions on the system of holes, for example, if the vertices of the holes form a discrete point system where the distance between every pair of points is no less than some r, and there is no empty circle of radius greater than some R, or, that the sizes of the holes are uniformly bounded both from above and below, it is not obvious that the union of all subsets of holes supporting tension can be represented as the complement of a discrete set of non-overlapping polygons.

Conjecture 2.5. Let $\mathscr{H}$ be an infinite discrete system of convex polygons in $\mathbb{R}^2$. Then the union of all subsets of $\mathbb{R}^2 \setminus H$ supporting tension can be represented as the complement of a discrete set of edge-disjoint convex polygons.

Let us summarize the implications of the existence of tension in the complement of the holes. They directly follow from (non-trivial) Proposition 1.11 the first part of which can be derived from the results of Connelly[(12)] and Connelly and Whiteley.[(14)]

Proposition 2.6. Let $\mathscr{H}$ be a discrete collection of convex open polygons in $\mathbb{R}^2$ possibly overlapping. If $\mathbb{R}^2 \setminus H$ supports tension, then the 1-skeleton of any triangulation of $\mathbb{R}^2 \setminus H$ is globally pseudorigid and prestress stable in $\mathbb{R}^3$.

There are also interesting connections between our model and convex geometry that, in its original form, are due to Maxwell[20, 21] and Cremona;[22] they are outlined in Menshikov *et al.*[4]

2.2. Bootstrap Percolation of Convex Defects

Menshikov *et al.*[4] assume that holes are associated with the nodes of a Poisson point process on $\mathbb{R}^2$. They show that if the "centers" of the holes are distributed in $\mathbb{R}^2$ according to a Poisson law and their shapes are i.i.d., tension disappears on all of $\mathbb{R}^2$ a.s. In fact, this result follows from a more general theorem of the authors on the behavior of iterated convex hulls of connected subsets of $\mathbb{R}^d$, where the initial configuration of subsets is distributed according to a Poisson law, and the shapes of the elements of the original configuration are independent of this Poisson distribution. For the latter problem they establish the existence of a critical threshold in terms of the number of iterated convex hull operations required for covering all of $\mathbb{R}^d$. Below we give a short account of their results.

Definition 2.7. A hole (f-hole) centered at $p \in \mathbb{R}^d$ is a region

$$H(p, f) = \left\{ p + f\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)\mathbf{x} \;\middle|\; \|\mathbf{x}\| \leqslant 1 \right\}$$

where f is a continuous positive function defined on a unit $(d-1)$-sphere.

Therefore, each hole is completely described by a couple (p, f), where p is the center of the hole and f is a continuous function on a unit sphere that determines the shape of the hole boundary. Consider a d-dimensional Poisson point process with rate λ. Let $Y = Y(\omega)$ be the collection of nodes of some realization ω of the process. Each node $y \in Y(\omega)$ is the center of a hole $H(y, f_y)$, where function f_y is positive and continuous. Let μ be a probability measure on some subset of positive continuous functions on the unit $(d-1)$-sphere. Suppose that for each y the function f_y is chosen from a distribution μ independently of the other functions and the configuration ω. Therefore, the holes $H(y, f_y)$ are i.i.d..

Definition 2.8. Let $\mathscr{H}$ be a set of holes. Elements of $\mathscr{H}$ are called defects of 0-th generation.

Definition 2.9. A connectivity component (understood topologically) of defects of the kth generation is referred to as a k-cluster.

Definition 2.10. A defect of the $(k+1)$th generation is the convex hull of a $k-$cluster.

Lemma 2.11 (Menshikov *et al.*[4]). Let our membrane M be all of $\mathbb{R}^2$. Then tension vanishes on a defect of any generation.

The following theorem from Menshikov *et al.*[4] is the main result for the independent Poisson model of tension percolation.

Theorem 2.12. For any distribution μ and any $\lambda > 0$ there exists a non-random non-negative integer $N = N(\mu, \lambda)$ such that N-cluster coincides with $\mathbb{R}^d$ a.s.

In this paper we establish a similar result for tension percolation on a triangular lattice. There are strong parallels between the continuous and the lattice models. The probability $1-p$ of independent edge deletion plays the role of the Poisson density λ. The number N_{cr} of applications of local rules (see Theorem 4.2) required to eliminate all the infinite connected components in the triangular lattice $\mathbb{T}_p$ is, in a way, similar to the number $N = N(\mu, \lambda)$ from the above theorem. N_{cr} and $N = N(\mu, \lambda)$ are both referred to as the *destruction* time.

3. TRIANGULAR LATTICE MODELS

We consider a regular or slightly perturbed triangular lattice $\mathbb{T}$ on the plane where each edge is removed independently with probability $1-p$, $p > 0$. Is there a critical value $p_c < 1$, such that for $p > p_c$ there is an infinite spider web subgraph a.s.? We show that for any $p < 1$ there is no spider web subgraph a.s. Thus, no non-trivial p_c exists. Our percolation model is related to so-called "bootstrap percolation" introduced on trees by Chalupa *et al.*[23] and, later, on d-dimensional lattices by Kogut and Leath.[24] In these models, points are independently occupied with a low density and the resulting configuration is taken as the initial state for dynamics based on some collection of local rules, in which the occupation status of a point is updated according to the configuration of its neighbors. van Enter[25] conducted a rigorous analysis of these models (see also Aizenman and Lebowitz[26]). For a review of bootstrap percolation models see Adler.[27] For the latest results on bootstrap percolation see Dehghanpour and Schonmann.[28]

Consider the affine plane $\mathbb{R}^2$ and two vectors $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$ with coordinates $(1, 0)$ and $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ respectfully . Also, set $\vec{\mathbf{e}}_3 = \vec{\mathbf{e}}_2 - \vec{\mathbf{e}}_1$. The regular triangular

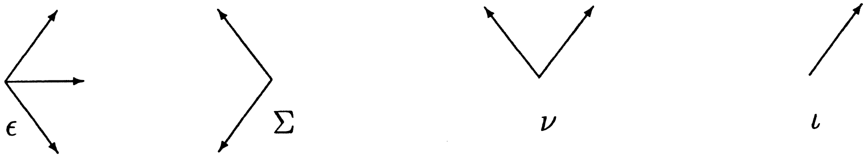

Fig. 10. Local removal rules.

lattice $\mathbb{T}$ is a framework whose vertex set is the collection of all points with coordinates $V(\mathbb{T}) = \{i\vec{\mathbf{e}}_1 + j\vec{\mathbf{e}}_2 \mid (i, j) \in \mathbb{Z}^2\}$, and whose edge set $E(\mathbb{T})$ consists of all edges between vertices $\mathbf{a}, \mathbf{b} \in V(\mathbb{T})$ such that $\mathbf{a} - \mathbf{b} = \vec{\mathbf{e}}_{\mathrm{k}}$ or $\mathbf{a} - \mathbf{b} = -\vec{\mathbf{e}}_{\mathrm{k}}$ for $k = 1$, 2 or 3. Let us denote an edge between a and b by $(\mathbf{a}, \mathbf{b})$.

Suppose some edge have been removed from $\mathbb{T}$. Denote the resulting lattice by $\mathbb{T}'$. By Proposition 1.4 edges in configurations congruent to those depicted in Fig. 10 cannot support tension. We call configurations in Fig. 10 ϵ-, Σ-, ν-, and ι-configurations respectfully, and refer to any such configuration as *relaxed.* By Proposition 1.5 edge $(\mathbf{va})$ and edges $(\mathbf{wa})$ and $(\mathbf{wb})$ in Fig. 11 cannot support tension. We call such edges *legs* in λ- and π-configurations. We refer to λ- and π-configurations as *partially relaxed.* Therefore, if $\mathbb{T}'$ contains a spider web as a subgraph, this spider web does not have edges in configuration depicted in Fig. 10 and edges that are legs in λ- or π-configurations.

Assume we have an infinite parallel Turing machine that can operate on the stars of the vertices of an infinite (but locally finite) grid; the machine works on all stars simultaneously. Once the machine sees a star where edges form one of the configurations congruent to those depicted in Figs. 10 or 11 (ϵ, Σ, ν, ι, λ, π), it removes all the edges that cannot support tension. The machine proceeds for as long as there are edges that can be removed using the local rules given by Figs. 10 and 11.

In Section 4 we show that if the initial lattice $\mathbb{T}_p$ was obtained from $\mathbb{T}$ as the result of the independent edge removal with probability $1-p$, the

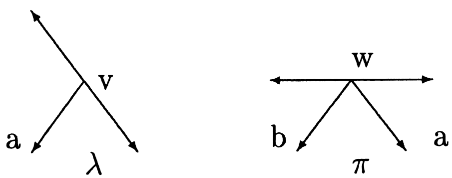

Fig. 11. Local partial removal rules.

parallel machine operating on the grid requires only a finite number of steps to turn $\mathbb{T}_p$ into a graph with no infinite connected components.

Definition 3.1. A framework $G(E, V, V_0; \mathbf{p})$ is said to support tension if it has a sub-framework which is a *spider web*.

Our main result is

Theorem 3.2. For any $p<1$ the lattice $\mathbb{T}_p$ obtained from the regular triangular lattice $\mathbb{T}$ with edge length 1 as the result of the independent edge removal with probability $1-p>0$ cannot support tension almost surely. This is also true for any perturbation of $\mathbb{T}_p$ such that the vertices are not farther away from the their original positions more than $1/4$.

However, first we want to prove

Lemma 3.3. With a *positive* probability $\mathbb{T}_p$ cannot support tension.

Proof of the Lemma. By Proposition 1.4 an edge incident to a vertex whose star is congruent to one of the stars depicted in Fig. 10 cannot support tension. Therefore, the lattice $\mathbb{T}_p$ can support tension if and only if the lattice $\mathbb{T}_p(1)$ obtained from $\mathbb{T}_p$ by removing all edges in such relaxed configurations can support tension. We call these edges *implicitly* removed, as opposed to *initially* removed edges, that is, $E(\mathbb{T}\setminus\mathbb{T}_p)$. Similarly, we construct the lattice $\mathbb{T}_p(2)$ by removing all edges from $\mathbb{T}_p(1)$ in configurations congruent to the ones in Fig. 10. In the same manner we define lattices $\mathbb{T}_p(3)$, $\mathbb{T}_p(4)$, ..., etc. Notice, that if $\mathbb{T}_p(n+1)\equiv\mathbb{T}_p(n)$ for some n, then $\mathbb{T}_p(n+k)\equiv\mathbb{T}_p(n)$ for any positive integer k.

Let k be a positive integer and $\mathscr{H}(k)$ be a regular hexagon centered at the origin with a side of length k, i.e., the hexagon with the vertices $k\vec{\mathbf{e}}_1$, $k\vec{\mathbf{e}}_2$, $k\vec{\mathbf{e}}_3$, $(-k)\,\vec{\mathbf{e}}_1$, $(-k)\,\vec{\mathbf{e}}_2$ *and* $(-k)\,\vec{\mathbf{e}}_3$. Let $F(k)$ denote the event "all interior edges of $\mathscr{H}(k)$ have been, possibly implicitly, removed from $\mathbb{T}_p(k)$ for some k." It is obvious from geometric observations that for any $k_0<k$

$$\mathsf{P}\left(F(k+1)\,\middle|\,\bigcap_{i=k_0}^{k} F(i)\right)=\mathsf{P}(F(k+1)\mid F(k)).$$

Let us show that

$$\mathsf{P}\left(\bigcap_{i=k_0+1}^{\infty} F(i)\,\middle|\,F(k_0)\right)=\prod_{k=k_0}^{\infty} P(F(k+1)\mid F(k)), \tag{3}$$

Indeed, for $k > k_0$

$$\mathsf{P}\left(\bigcap_{i=k_0+1}^{k} F(i) \mid F(k_0)\right) = \frac{\mathsf{P}(F(k) \mid \bigcap_{i=k_0}^{k-1} F(i))\, \mathsf{P}(\bigcap_{i=k_0}^{k-1} F(i))}{\mathsf{P}(F(k_0))}$$

$$= \mathsf{P}(F(k) \mid F(k-1))\, \mathsf{P}\left(\bigcap_{i=k_0+1}^{k-1} F(i) \mid F(k_0)\right)$$

$$= \cdots = \prod_{i=k_0}^{k} \mathsf{P}(F(i+1) \mid F(i))$$

Letting $k \to \infty$ proves (3).

We will show that for large k the probability of the event $F(k+1) \mid F(k)$ is greater than $1-\gamma_k$, for some sequence $\{\gamma_k\}$, such that $\sum_{(k=k_0)}^{(\infty)} \gamma_k < \infty$. This yields

$$\mathsf{P}\left(\bigcap_{i=k_0+1}^{\infty} F(i) \mid F(k_0)\right) \geqslant \prod_{k=k_0}^{\infty} (1-\gamma_k) > 0,$$

which, in turn, proves the Lemma, since $P(F(k_0)) > 0$ for any fixed k_0 and positive p.

Indeed, the probability that on each of the six sides of $\mathscr{H}(k)$ at least one edge has been initially removed is

$$(1-p^k)^6 = (1-e^{-\alpha k})^6 > 1-6e^{-\alpha k}$$

where $\alpha = -\log p > 0$. Now, pick k_0 so large that $1-6e^{-\alpha k}$ is positive as soon as $k \geqslant k_0$. Set $\gamma_k = e^{-\alpha k}$. Then $\sum \gamma_k$ is, indeed, finite. Meanwhile, as one can conclude upon studying Fig. 12, whenever there are no edges inside $\mathscr{H}(k)$, and at least one edge is removed on each side of it, an incremental application of the removal rules depicted in Fig. 10 will eventually, (in a number of steps not exceeding k), delete all edges inside $\mathscr{H}(k+1)$. If an edge is missing on one of the sides of $\mathscr{H}(k)$, then there are two vertices v_1 and v_2 on this side whose stars look like some two stars in the Fig. 10. According to the removal rules, the stars of v_1 and v_2 should be removed. Following this procedure in both directions along the side of $\mathscr{H}(k)$ we eventually remove all the edges on this side.

Therefore, with a positive probability the event $F(k)$ implies that all the edges of our lattice are eventually removed.

Let us discuss the case when the coordinates of the vertices of $\mathbb{T}_p$ have been perturbed. Assume $\mathbb{T}$ is regular and the edges of $\mathbb{T}$ are of length 1. Let $\mathbb{T}'_p$ be a framework whose graph is isomorphic to that of $\mathbb{T}_p$, and whose vertices are located at the distance of at most $1/4$ from the corresponding

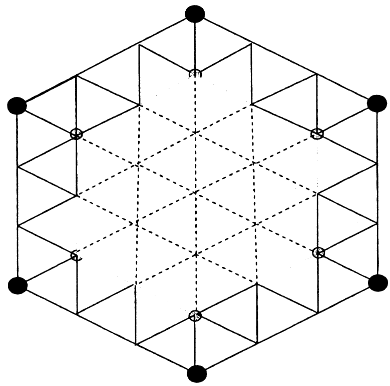

Fig. 12. Typical propagation of a regular hexagon. Solid lines are remaining edges, dotted lines are removed ones. Dark circles are vertices of $\mathscr{H}(k)$, and grey circles are vertices of $\mathscr{H}(k-1)$.

vertices of $\mathbb{T}_p$. In other words, $\mathbb{T}'_p$ is a perturbed version of $\mathbb{T}_p$, and the displacement of each vertex is limited by $1/4$. We need only local removal rules that are depicted in Fig. 10. The angle between any two (directed) edges in any of these four configurations cannot become greater than Π if the vertices are not further than $1/4$ from their original positions. Thus, the edges in these configurations cannot bear tension and should be removed according to our local rules. Therefore all arguments of the proof work for the perturbed lattice $\mathbb{T}'_p$. ∎

Notice that in our model an empty hexagon propagating to infinity plays the role of a "critical droplet," sometimes called "Straley void." Before returning to our main theorem we would like to make a few important observations. Below, we will refer to the process described in the above proof as "hexagon propagation." We will make use of the following definition.

Definition 3.4. We say that a sequence of planar frameworks $L(n)$ *eventually disappears* and write $L(n) \to \varnothing$, if for any fixed bounded subset A of the plane there exists $N > 0$ such that $L(n) \cap A = \varnothing$ for all $n \geqslant N$.

Therefore, the above lemma immediately implies

Corollary 3.5. With a positive probability, $\mathbb{T}_p(n) \to \varnothing$. Moreover, conditioned on the event $R_k =$ "all edges are initially removed in $\mathscr{H}(k)$,"

$$\mathsf{P}(\mathbb{T}_p(n) \to \varnothing \mid R_k) \to 1$$

as $k \to \infty$.

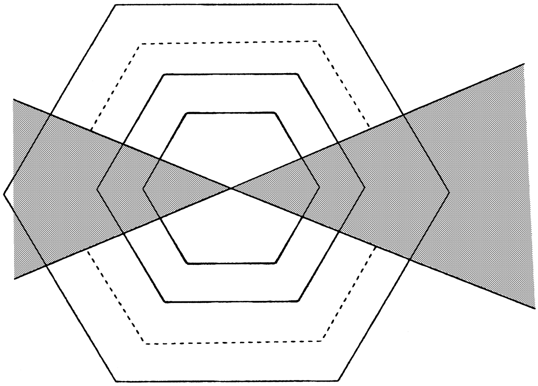

Fig. 13. Hexagon propagation avoiding two angles.

We would like to make another observation about the proof of Lemma 3.3. Suppose the interior of $\mathscr{H}(k)$ is empty. Evidently, to remove all edges from $\mathscr{H}(k+1)$ using local removal rules described above we need that at least one edge is absent (initially removed) on each side of $\mathscr{H}(k)$. Suppose we are not allowed to look for such initially removed edges in the planar cones (angles) defined by inequalities $|\varphi| \leqslant 30°$ and $|\varphi - 180°| \leqslant 30°$, in the polar coordinate system (ρ, φ) (see Fig. 13). It is not hard to check that the arguments of the proof of Lemma 3.3 can be carried through virtually unchanged. Thus we have

Corollary 3.6. Independently of the initial configuration inside the above mentioned cones

$$\mathsf{P}(\mathbb{T}_p(n) \to \varnothing \mid R(k)) \to 1$$

as $k \to \infty$.

Notice that the above corollaries hold for a perturbed version of $\mathbb{T}_p$ where the vertices moved the distances not exceeding $1/4$. Let us return to our main statement.

Proof of Theorem 3.2. Fix $\epsilon > 0$. By Corollary 3.6, there is N large enough such that, if each edge in $\mathscr{H}(N)$ has been removed, the probability that $\mathbb{T}_p(n) \to \varnothing$ is greater than $1 - \epsilon/2$, regardless of the configuration inside the two cones. Let $q = q(N)$ be the probability that all edges inside

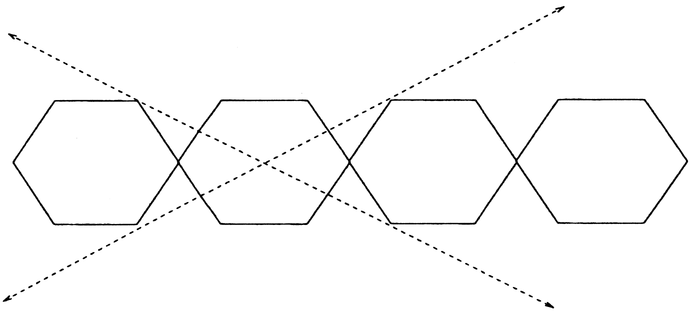

Fig. 14. M hexagons; with probability $1-\epsilon/2$ all edges are removed in at least one of them.

$\mathscr{H}(N)$ have been initially removed. Obviously, $q>0$ for any positive p. There is a positive integer M such that

$$1-(1-q)^M > 1-\epsilon/2.$$

Consider M non-overlapping hexagons $\mathscr{H}_i(N)$, $i=0,\ldots, M-1$, of size N along the horizontal axis with the centers at $\mathbf{0}$, $N\vec{\mathbf{e}}_1$, $2N\vec{\mathbf{e}}_1,\ldots$, $(M-1)\, N\vec{\mathbf{e}}_1$ (see Fig. 14). Notice, that each of the hexagons lies fully inside $\pm 30°$ angles for all the others; therefore, due to the symmetry and space homogeneity, there is $N>0$ such that each $\mathscr{H}_i(N)$ propagates to infinity in the way described in the proof of Lemma 3.3 (with an angular restriction of Corollary 3.6) with probability at least $1-\epsilon/2$ *independently* of the initial configuration inside the others. Thus, the probability that inside of at least one of the M hexagons all the edges have been initially removed, *and* this hexagon will propagate to infinity is greater than

$$(1-\epsilon/2)^2 \geqslant 1-\epsilon.$$

Now, recall the definition of tension. A framework supports tension if there is a subgraph of this framework that can bear an equilibrium tension. The local rules cull only those edges that cannot support tension. The arguments above show that eventually *all* edges are bound to be removed with probability at least $1-\epsilon$. Since $\epsilon>0$ is arbitrary, Theorem 3.2 holds. ∎

Studies of rigidity percolation (Jacobs *et al.*,[15, 16, 29] and Holroyd[8]) show that the behavior of a regular triangular lattice may differ from the behavior of a generic triangular lattice. A generic lattice in a strong sense is a realization of a graph in $\mathbb{R}^d$ where the dimension of the space of stresses of any finite subgraph of the lattice is minimal. All theorems and lemmas in this section hold not only for a regular triangular lattice $\mathbb{T}$, but also for any generic triangular lattice obtained from $\mathbb{T}$ by a sufficiently small perturbation, for we essentially need only three removal rules: the ν-rule, the Σ-rule, the ϵ-rule, and the ι-rule, which are "robust" to such perturbations. (see Lemma 3.3 and Fig. 13). Of course, our tension percolation problem for a perturbed triangular lattice makes sense only if there are perturbations of the regular lattice preserving the property of the lattice to support an equilibrium tension. It follows from the results of Barany and Dolbilin[30] or Connelly[11] on the uniform stability of sphere packings that there is $\epsilon > 0$ such that any ϵ-perturbation of the regular triangular lattice supports an equilibrium tension (see also Bezdek *et al.*[7]).

We suspect that all our results hold for a larger class of generic triangular lattices, although our method cannot be applied straightforwardly to the case of an arbitrary generic triangular lattice, because a perturbation can turn a relaxed configuration into a non-relaxed configuration (see Fig. 2).

A general tension percolation problem can be stated as follows. Let G be an infinite framework in $\mathbb{R}^d$ with discrete vertex set. Remove each edge with probability $1-p$ independently of the other edges, and denote the resulting graph by G^p. What is the infinum of p's such that G^p supports tension a.s.? We call this number the critical probability of tension percolation. We have a general conjecture about tension percolation on planar graphs. To formulate this conjecture we need to introduce the notion of directional spectrum of a framework. By the direction of a line on the plane we understand the angle this line forms with, say, the horizontal axis. If G is a framework on the plane, the set of directions defined by the edges of G is called the *directional spectrum* of G. The edge set of a framework G is said to have the (l, L) property if the edge lengths of G are bounded from below by some $l > 0$ and from above by some $L > 0$.

Conjecture 3.7. Let $G = (E, V, V_0)$ be an infinite framework on the plane realized without self-intersections. Suppose the directional spectrum of G is finite, and E has the (l, L) property. Then the critical probability of tension percolation is 1.

The notion of an (r, R) point system is widely used in discrete geometry and mathematical crystallography. A point set V is called an (r, R)-system, or a Delaunay system, if

(1) for any point $\mathbf{v} \in V$ the ball of radius r centered at $\mathbf{v}$ does not contain any other vertices of V, and

(2) any ball of radius R contains at least one point of V.

Notice, that for a graph with a finite directional spectrum the (l, L)-property of the edge set is equivalent to the (r, R)-property of the vertex set.

4. FINITE TIME OF RELAXATION

Assume that it takes one unit of time for an infinite parallel Turing machine to remove all the configurations of edges congruent to those depicted in Fig. 10. Thus, the lattice $\mathbb{T}_p$ is transformed to $\mathbb{T}_p(n)$ by time n. Let us call this process the *relaxation of tension on* $\mathbb{T}_p$, and say that tension has been completely lost if there is no infinite connected component of non-removed edges on the lattice. We shall show that the complete relaxation of tension occurs in a finite time a.s. We shall also show that there is a non-random time $N \geqslant 1$ such that $\mathbb{T}_p(N)$ has no infinite connected components a.s., but $\mathbb{T}_p(N-1)$ has an infinite component a.s. (by convention, we set $\mathbb{T}_p(0) = \mathbb{T}_p$).

Lemma 4.1. The event $T(p, N) :=$ "$\mathbb{T}_p(n)$ has an infinite connected subgraph" is a tail event.

Proof. We need to show that this event does not depend on the state of any finite subset of $\mathbb{T}$. Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two subgraphs of our triangular lattice $\mathbb{T}$ such that $\mathbb{T}_1$ can be obtained from $\mathbb{T}_2$ by adding and/or removing only a finite number of edges. Let E be those edges of $\mathbb{T}_1$ that are absent in $\mathbb{T}_2$. Denote by $\mathbb{T}_1(n)$ and $\mathbb{T}_2(n)$ the results of n iterated applications of the local rules to $\mathbb{T}_1$ and $\mathbb{T}_2$. Suppose $\mathbb{T}_1(n)$ has an infinite connected component C. If $\mathbb{T}_2(n)$ has no infinite connected component, $\mathbb{T}_2(n)$ differs from $\mathbb{T}_1(n)$ at infinitely many places. An edge e of C can be absent from $\mathbb{T}_2(n)$, only if there is an edge path on $\mathbb{T}$ of length no greater than n connecting a vertex of e to one of the vertices of E. Thus, only those edges of C can be missing from $\mathbb{T}_2$ that lie at distance no greater than n from the vertex set of E. The number of such edges is finite. Therefore $\mathbb{T}_2$ contains an infinite connected component of C which is, in turn, is an infinite connected subgraph of $\mathbb{T}$. ▮

Theorem 4.2. There is a non-random number N_{cr} such that $\mathbb{T}_p(N_{cr})$ is a union of finite disjoint graphs a.s., but $\mathbb{T}_p(N_{cr}-1)$ has an infinite connected component a.s. (and $\mathbb{T}_p(0) \equiv \mathbb{T}$ as before).

Proof. From the above lemma and Kolmogorov's $0-1$ law it follows that $\mathsf{P}(T(p, n))$ is either zero or one. Moreover, this probability is non-increasing as n grows, and $\mathsf{P}(T(p, 0)) = 1$. Therefore, either there is N_{cr}

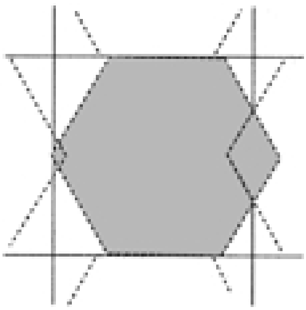

Fig. 15. Hexagon propagation inside a box.

such that $\mathsf{P}(T(p, N_{cr}-1)) = 1$ and $\mathsf{P}(T(p, N_{cr})) = 0$, or $\mathsf{P}(T(p, n)) = 1$ for all n. To rule out the second possibility, it suffices to show that there is positive integer N such that $\mathbb{T}_p(N)$ has no infinite connected component a.s.

The idea of the proof is based on Theorem 3.2. Let N and M are the same as in the proof of Theorem 3.2. Pick $\epsilon < \frac{1}{10}$ and N and M corresponding to this ϵ. Consider a partition of the plane into the boxes with the side length $S = 13(M-1)\, N > (4\sqrt{3}+6)(M-1)\, N$. Assume one of the boxes—call it B_0—is centered at $\mathbf{0}$. In this box consider hexagons with the side N centered at $\mathbf{0}$, $N\vec{\mathbf{e}}_1$, $2N\vec{\mathbf{e}}_1$,..., $(M-1)\, N\vec{\mathbf{e}}_1$.

We call box B_0 *open* if (1) one of these hexagons has all the edges removed, (2) using the procedure of implied edge removal as described by Lemma 3.3 and avoiding $\pm 30°$ cones, it will grow till its upper and lower sides coincide with those of the box B_0 and (3) one of the edges on its upper side with the X-coordinate between 0 and $(M-1)\, N$ has been initially removed (see Figs. 14 and 15).

Following the line of arguments in Theorem 3.2, we can conclude that the probability that B_0 is open can be made greater than 0.9 (however, we might need to have N quite large). The same is true about the other boxes of the tiling $\{B_0 + i\vec{\mathbf{e}}_1 + j \times \vec{(1, 0)} \mid (i, j) \in \mathbb{Z}^2\}$. Moreover, both vertical and horizontal neighbors are open independently, since they "look for" different initially removed edges (this is because we ignore the interior of the cones described between Corollaries 3.5 and 3.6). Therefore, all the boxes are open independently of each other. Besides, if two neighboring (on a side) boxes are open, their interior areas, where edges are removed, are connected.

Now, let us couple the boxes with the vertices in the site percolation model on $\mathbb{Z}^2$, where each site is open with probability 0.9 and closed otherwise. There is a unique open cluster of open sites and no infinite cluster of closed sites (e.g., see Aizenman *et al.*[31] or Grimmett[10]). Therefore, each cluster of closed sites is surrounded by a finite contour of open site. Geometrically, for our triangular lattice, it implies that each connected component of non-removed edges is surrounded by a contour of removed ones, and therefore each such component is finite. Thus, after N (or even less) iterations $\mathbb{T}_p(N)$ has no infinite connected component a.s. ∎

5. DISCUSSION: TENSION PERCOLATION ON 3D LATTICES AND FINITE SUBGRAPHS OF 2D LATTICES

Let us sketch how tension percolation on a 3-dimensional "triangular" lattice relates to the spider web properties of finite subgraphs of a 2-dimensional triangular lattice. To introduce a three-dimensional analog of $\mathbb{T}$ we need to enlist the notion of *point lattice*. Recall that a point lattice is the set of all points in the affine space $\mathbb{R}^d$ that can be represented as integral linear combinations of the vectors of a fixed basis of $\mathbb{R}^d$. A face-centered cubic lattice of points—*fcc* lattice—is constructed by adding the centers of all the facets of a standard tiling by cubes to the vertex set of this tiling, lattice $\mathbb{Z}^3$. This lattice is a natural 3-dimensional generalization of the hexagonal point lattice (i.e., the vertex set of what we call in this paper the regular triangular lattice), since it is generated by the edge set of a regular 3-simplex. Denote by $\mathbb{T}^3$ the graph whose vertex set is the *fcc* lattice, and whose edge set consists of all unordered pairs of vertices $(\mathbf{a}, \mathbf{b})$ such that $\mathbf{a}-\mathbf{b}$ is the shortest vector of our *fcc* lattice. Remove each edge independently with probability $1-p$ and denote the resulting lattice by $\mathbb{T}^3_p$. For what values of p does the modified lattice support tension with a positive probability?

Suppose we want to approach this problem in the same way we approached the 2-dimensional problem. Here, instead of a propagating hexagon we have a propagating 3-polytope (see Lemma 3.3). Notice, that regular triangular and square lattice are the only types of 2-dimensional (edge) sublattices of $\mathbb{T}^3$.

We call P a lattice polytope if all its faces lie on periodic connected subgraphs of $\mathbb{T}^3$; note that our definition of a lattice polytope differs from the standard definition of a lattice polytope used in the theory of lattice points. Thus, the facets of a lattice 3-polytope P can only be of two sorts: lying on a square sublattice and lying on a triangular sublattice (here *sublattices* are understood in the sense of periodic graphs). The geometry of a particular facet as well as the geometry of P are not so important, since there are only a finite number of lattice polytopes in $\mathbb{T}^3$ up to affine equivalence. From this remark it becomes clear that, in principal, tension percolation on $\mathbb{T}^3$ is no different from tension percolation on any periodic graph in $\mathbb{R}^3$ with triangular planar subgraphs. Now, let us compare the hexagon propagation and the polytope propagation (of course, we work with the graph that this polytope cuts out of $\mathbb{T}_p^3$, rather than with the polytope per se). If a side of the propagating hexagon misses an edge, the entire side has to go; however, a missing edge on a facet of the propagating polytope is not enough to conclude that the rest of the facet (with fixed boundary) is not able to bear tension.

Let P be a propagating (convex) polytope in $\mathbb{T}$ of some fixed shape. That is, we consider nested copies of P in $\mathbb{T}_p^3$ propagating to infinity, just like we considered propagating hexagons in $\mathbb{T}_p$ (see Section 4). Denote an infinite sequence of such nested polytopes by $P(n)$, where n is the linear size of the $P(n)$. Let F be a face of P, and let $F(n)$ be the corresponding sequence of faces of $P(n)$. $F(n)$ is a finite subgraph of $\mathbb{T}_p$ bounded by a convex polygon. Denote by $P_p(F, n)$ the probability that after each edge has been removed independently from F with probability $1-p$ the resulting graph (with fixed boundary) has a spider web subgraph. If $\sum_n P_p(F, n)$ converges, the arguments of Lemma 3.3 and Theorem 3.2 work, and $\mathbb{T}_p^3$, $p<1$ cannot support tension a.s. Notice that if F has the square type, $P_p(F, n)$ obviously converges, since each term in the series is bounded from above by $2np^n$. Of course, even if no analog of Lemma 3.3 can be proved in dimension 3, the critical probability of tension percolation for $\mathbb{T}^3$ may well be 1. We end up with the following problem.

Problem 5.1. Is it true that the critical probability of tension percolation for $\mathbb{T}^3$ is less than 1?

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